ON THE CONVEXITY OF THE JOINT ESSENTIAL NUMERICAL RANGES

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Abstract

The concept of essential numerical range of an operator was defined and studied by Stampfli and Williams in 1972. In this paper, we consider the joint essential numerical range of an m-tuple of operators $T = (T_1, \ldots, T_m)$ in $B(X)$ and show that the convexity of the classical numerical range also hold for the joint essential numerical range.

1. Introduction

$B(X)$ will denote the algebra of (bounded) linear operators acting on complex Hilbert space X.

The joint Numerical range of $T = (T_1, \ldots, T_m) \in S(X)^m$ is defined as

$$W_m(T) = \{(\langle T_1x, x\rangle, \ldots, \langle T_mx, x\rangle): x \in X, \langle x, x\rangle = 1\}$$

The ideal of all compact operators in $B(X)$ be denoted by $\mathcal{K}(X)$. The joint essential numerical range of $T \in S(X)^m$ is defined as

$$W_{e_m}(T) = \{\overline{W_m(T+K)} : K = (K_1, \ldots, K_m) \in \mathcal{K}(X)\}.$$

Studying convexity of $W_m(T)$ and $\overline{W_m(T+K)}$, researchers concluded that $W_m(T)$ is convex for $m = 1$ and not convex in general for $m \geq 2$ while $\overline{W_m(T+K)}$ is non-convex, see [1, 2, 3, 6, 7]. It is thus unexpected for the set $W_{e_m}(T)$ to be convex since it is an intersection of non-convex sets. One of the objects of this paper is to show that the joint essential numerical range is always convex. The essential numerical range, $W_e(T)$, for a single operator was studied in [5] and its equivalent definitions given. Generalising these results $W_{e_m}(T)$ is also defined by $W_{e_m}(T) = \{r \in \mathbb{C}^m : \langle Tx_n, x_n\rangle \longrightarrow r_k, x_n \longrightarrow 0\ weakly; 1 \leq k \leq m\}$.

Theorem 2.1. *Let* $T = (T_1, \dots, T_m)$ *be an m-tuple of operators on X. For a point* $\{r = (r_1, \dots, r_m) \in \mathbb{C}^m\}$, *there exists an orthonormal sequence* $\{x_n\}_{n=1}^{\infty} \in X$ *such that if and only if* $r \in W_{e_m}(T)$

$$\langle Tx_n, x_n\rangle \longrightarrow r_k; 1 \leq k \leq m.$$

Proof. Assume that for a point $r = (r_1, \dots, r_m) \in \mathbb{C}^m$, there exists an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that

$\langle Tx_n, x_n\rangle \longrightarrow r_k; 1 \leq k \leq m.$

But every orthonormal sequence $\{x_n\}$ converges weakly to zero and $\|x_n\| = 1$. Thus $r \in W_{e_m}(T)$.

Conversely, let $r \in W_{e_m}(T)$ and show that there exists an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that

$$\langle Tx_n, x_n\rangle \longrightarrow r_k; 1 \leq k \leq m.$$

Since $r \in W_{e_m}(T)$ there is a sequence $\{x_n\}$ of vectors such that

$\langle Tx_n, x_n\rangle \longrightarrow r_k, \|x_n\| = 1, x_n \longrightarrow 0\ weakly; 1 \leq k \leq m.$

Suppose we have chosen the set $\{x_1, \dots, x_n\}$ satisfying $|\langle Tz_n, z_n\rangle - r| < \frac{1}{i}, \forall\ i.$

Let $\mathcal{M}$ be the subspace spanned by $x_1, \dots, x_n$ and P be the projection onto

M. Then we have $\|Px_n\| \to 0$ as $n \to \infty$. Let

$$z_n = \|(I-P)x_n\|^{-1}\left((I-P)x_n\right).$$

We have

$$Tz_n = \|(I-P)x_n\|^{-1}(T(I-P)x_n)\,.$$

This gives

$$\langle Tz_n, z_n\rangle = \langle \|(I-P)x_n\|^{-1}(T(I-P)x_n), \|(I-P)x_n\|^{-1}(T(I-P)x_n)\rangle =$$

$\|(I-P)x_n\|^{-2}\{\langle Tx_n, x_n\rangle - \langle TPx_n, Px_n\rangle - \langle Tx_n, x_n\rangle + \langle TPx_n, Px_n\rangle\} \longrightarrow r_k$ as $n \to \infty$.

We choose n large enough such that

$$|\langle Tz_n, z_n\rangle - r| < \frac{1}{n+1}.$$

If we let $z_n = x_n + 1$ we get

$$|\langle Tx_{n+1}, x_{n+1}\rangle - r| < \frac{1}{n+1}.$$

We remind the reader that

A subset C of a linear space M is convex if $\forall\, x; y \in C$ the segment joining x and y is contained in C, that is,

$tx + (1-t)y \in C \;\forall\, t \in [0; 1]$.

A set S is starshaped if $\exists\, y \in S$ such that $\forall\, x \in S$ the segment joining x and y is contained in S, that is $\lambda x + (1 - \lambda\_)y \in S \;\forall\, \lambda \in [0; 1]$.

A point $y \in S$ is a star center of S if there is a point $x \in S$ such that the segment joining x and y is contained in S:

Starshapedness is related to convexity in that a convex set is starshaped with all its points being star centers.

Theorem 2.2. *Suppose $T = (T_1, \dots, T_m) \in B(X)$: Then $W_{e_m}(T)$ is nonempty, compact and each element $r \in W_{e_m}(T)$. is a star center of $\overline{W_m T}$. Moreover, $W_{e_m}(T)$ is convex.*

Proof. First, we prove that $W_{e_m}(T)$ is nonempty. To do this, from Theorem 1.1, there exists an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that

$$\langle T x_n, x_n \rangle \longrightarrow r_k; 1 \leq k \leq m.$$

Thus the sequence $\{\langle T x_n, x_n \rangle\}_{n=1}^{\infty}$ is bounded. Choose a subsequence and assume that $\langle T x_n, x_n \rangle$ converges. Then $W_{e_m}(T)$ is nonempty.

The compactness of $W_{e_m}(T)$ can be seen right from its definition. The joint essential numerical range, $W_{e_m}(T)$ is defined as the intersection of all sets of the form $\overline{W_m(T + K)} : K \in \mathcal{K}(X)$ where $\mathcal{K}(X)$ denote the sets of compact operators in $B(X)$. Being an intersection of compact sets, the joint essential numerical range is also compact.

To prove that each element $r \in W_{e_m}(T)$ is a star center of $\overline{W_m(T)}$, it should be shown that $(1 - \lambda)p + \lambda r \in \overline{W_m T} : \lambda \in [0,1]$ where $r \in W_{e_m}(T)$ and $p \in \overline{W_m(T)}$. Assume without loss of generality that $\|T\| = 1$. Suppose $s \in \overline{W_m(T)}$ so that $s = \lambda r + (1 - \lambda)p$. Let $\{x_n\}$ and $\{e_n\}$ be orthonormal sequences in X such that

$r = \langle T x_n, x_n \rangle$, $p = \langle T e_n, e_n \rangle$ and

$\|x_n\| = \|e_n\| = 1.$

Then,

$$\begin{aligned} s &= \lambda \langle T x_n, x_n \rangle + (1 - \lambda) \langle T x_n, x_n \rangle \\ &= \langle T \sqrt{\lambda} x_n, \sqrt{\lambda} x_n \rangle + \langle T \sqrt{1 - \lambda} x_n, \sqrt{1 - \lambda} x_n \rangle \\ &= \langle \left(T \sqrt{\lambda} x_n + \sqrt{1 - \lambda} e_n \right), (\sqrt{\lambda} x_n + \sqrt{1 - \lambda} e_n) \rangle \end{aligned}$$

$$\left\|\sqrt{\lambda}x_n + \sqrt{1-\lambda}e_n\right\|^2 = \left(\left\|\sqrt{\lambda}x_n\right\|^2 + \left\|\sqrt{1-\lambda}e_n\right\|^2\right)$$

$$= \lambda\|x_n\|^2 + (1-\lambda)\|e_n\|^2$$

$$= \lambda + (1-\lambda) = 1$$

Thus, $(1-\lambda)r + \lambda p \in \overline{W_m(T)}$.

Convexity of $W_{e_m}(T)$ is proved by showing that $r, p \in W_{e_m}(T)$ and $\lambda \in [0,1], \lambda r + (1-\lambda)p \in W_{e_m}(T)$. Now, $r \in W_{e_m}(T) = r \in W_{e_m}(T+K)$ for every $K \in \mathcal{K}(X)$ and $p \in W_{e_m}(T) \subseteq \overline{W_m(T+K)}$. From Theorem above, $\lambda r + (1-\lambda)p \in \overline{W_m(T+K)}$. Hence $W_{e_m}(T)$ is convex.

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