EXPRESSION OF MOTION EQUATIONS FROM VECTOR FORM TO SPHERICAL COORDINATES

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ABSTRACT
Navier-Stokes equations are the equations of conservation of linear momentum. In this paper we express this type of equation from vector form to spherical co-ordinates. We refer to some equations in this paper from our first paper for the readability of this second paper.

INTRODUCTION

The general form of the equations for incompressible flow of Newtonian (constant viscosity) fluid is given by $\frac{\partial q}{\partial t} = -(\boldsymbol{q}.\boldsymbol{\nabla})\boldsymbol{q} + \upsilon\nabla^2\boldsymbol{q} - \frac{1}{\rho}\nabla P + \boldsymbol{g} + \boldsymbol{f}$

(1)

$\upsilon$is kinetic viscosity (constant) and is given by $\upsilon = \frac{\mu}{\rho}$, ρ is density (constant), $\boldsymbol{P}$ is pressure and $\boldsymbol{g}$is the gravitational force.

In the equation (9) above,

- $\frac{\partial q}{\partial t}$ – Acceleration term
- $(\boldsymbol{q}.\boldsymbol{\nabla})\boldsymbol{q}$ – is the advection term; the force exerted on the particles of the fluid by other particles of the fluid surrounding it
- $\upsilon\nabla^2\boldsymbol{q}$ – velocity diffusion terms; describes how the fluid motion is damped, highly viscous fluid e.g. honey stick together while low viscous fluid flow freely, e.g. air
- $\boldsymbol{\nabla}P$ - pressure term, fluids flow in the direction of largest change in pressure

From equation (2), $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ and $\boldsymbol{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$. Replacing this in equation (9) we obtain

$$\frac{\partial}{\partial t}(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) = -(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w).\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) +$$

$$v\nabla^2(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) - \frac{1}{\rho}\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)P + \rho(\mathbf{i}g_x + \mathbf{j}g_x + \mathbf{k}g_x) + (\mathbf{i}f_x + \mathbf{j}f_x + \mathbf{k}f_x)$$

(2)

In equation (10) the laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Replacing the laplacian above in equation (10), we obtain

$$\frac{\partial}{\partial t}(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) + v\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) - \frac{1}{\rho}\left(\mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z}\right)P + (\boldsymbol{i}g_x + \boldsymbol{j}g_x + \boldsymbol{k}g_x) + (\boldsymbol{i}f_x + \boldsymbol{j}f_x + \boldsymbol{k}f_x) \quad (3)$$

Collecting coefficients of, $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ together, this leads to equations in x, y and z directions respectively as follows;

$$\frac{\partial u}{\partial t} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u + v\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u + \frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + f_x \quad (4)$$

$$\frac{\partial v}{\partial t} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)v + v\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)v + \frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + f_y$$

$$\frac{\partial w}{\partial t} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)w + v\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)w + \frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + f_z$$

The above three equations are the Navier-Stokes equations in x, y and z components. In this project we will neglect the body forces. Therefore dropping the body forces and rearranging the equations we obtain;

x-component

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (5)$$

y-component

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{6}$$

z-component

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{7}$$

Here, υ is the coefficient of viscosity, ρ is the density of the fluid and $\mathbf{g}$ is the gravitational force.

MAIN RESULTS

EQUATIONS OF MOTION IN SPHERICAL COORDINATES

Finally, we shall convert the equations of motion in spherical coordinates. In this section however we will not convert the equations of motion in curvilinear coordinates since we have already done this in the previous paper. We will simply use what we have already derived.

The coordinates of a point P in spherical coordinates is given by the ordered pair, r, θ and $\emptyset$.

- r is the distance from the origin to the point P
- θ is the angle between the x −axis and the line from the origin to the point P
- $\emptyset$ is the angle between the z − axis and the line from the origin to the point P

The relationship between the cylindrical coordinates and the spherical coordinates is given by;

$$x=r\sin\phi\cos\theta,\quad y=r\sin\phi\sin\theta,\qquad z=r\cos\phi \tag{8}$$

Scale factors

In spherical coordinates, the scale factors are, h_r, h_θ and h_ϕ as shown in equation (26) are given by;

$$h_i^2=\left(\frac{\partial x}{\partial u_i}\right)^2+\left(\frac{\partial y}{\partial u_i}\right)^2+\left(\frac{\partial z}{\partial u_i}\right)^2 \tag{9}$$

Where $i=r,\ \theta$ and ϕ

$$h_r^{\,2} = \left(\frac{\partial x}{\partial u_r}\right)^2 + \left(\frac{\partial y}{\partial u_r}\right)^2 + \left(\frac{\partial z}{\partial u_r}\right)^2 \tag{10}$$

$$\begin{aligned} h_r^{\,2} &= \left(\frac{\partial}{\partial r} r\sin\phi\cos\theta\right)^2 + \left(\frac{\partial}{\partial r} r\sin\phi\sin\theta\right)^2 + \left(\frac{\partial}{\partial r} r\cos\phi\right)^2 \\ &= \sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi = \sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) + \cos^2\phi = 1 \end{aligned} \tag{11}$$

Similarly h_θ can be given by

$$h_\theta^{\,2} = \left(\frac{\partial x}{\partial u_\theta}\right)^2 + \left(\frac{\partial y}{\partial u_\theta}\right)^2 + \left(\frac{\partial z}{\partial u_\theta}\right)^2 \tag{12}$$

$$\begin{aligned} h_\theta^{\,2} &= \left(\frac{\partial}{\partial u_\theta} r\sin\phi\cos\theta\right)^2 + \left(\frac{\partial}{\partial u_\theta} r\sin\phi\sin\theta\right)^2 + \left(\frac{\partial}{\partial u_\theta} r\cos\phi\right)^2 \\ &= r^2\sin^2\phi\sin^2\theta + r^2\sin^2\phi\cos^2\theta = r^2\sin^2\phi\left(\sin^2\theta + \cos^2\theta\right) = r^2\sin^2\phi \end{aligned} \tag{13}$$

Similarly, h_ϕ can be expressed as

$$h_\phi^{\,2} = \left(\frac{\partial x}{\partial u_\phi}\right)^2 + \left(\frac{\partial y}{\partial u_\phi}\right)^2 + \left(\frac{\partial z}{\partial u_\phi}\right)^2 \tag{14}$$

$$\begin{aligned} h_\phi^{\,2} &= \left(\frac{\partial}{\partial u_\phi} \rho\sin\phi\cos\theta\right)^2 + \left(\frac{\partial}{\partial u_\phi} \rho\sin\phi\sin\theta\right)^2 + \left(\frac{\partial}{\partial u_\phi} \rho\cos\phi\right)^2 \\ &= r^2\cos^2\phi\cos^2\theta + r^2\cos^2\phi\sin^2\theta + r^2\sin^2\phi = r^2\cos^2\phi\left(\cos^2\theta + \sin^2\theta\right) + r^2\sin^2\phi \\ &= r^2\left(\cos^2\phi + \sin^2\phi\right) = r^2 \end{aligned} \tag{15}$$

Therefore the scale factors in spherical coordinates are given by

$$h_r^{\,2} = 1, \quad h_\theta = r\sin\phi, \text{ and } h_\phi = r \tag{16}$$

In spherical coordinates, the velocity vector $\boldsymbol{q}$ is given by,

$$q = u_r e_r + u_\theta e_\theta + u_\phi e_\phi \tag{17}$$

We will then express in spherical coordinates, the continuity equations and the momentum equation

Continuity equation

We will express the continuity equation in equation (7)$\nabla.\boldsymbol{q}$ = 0 in curvilinear coordinates. From equation (60), we have;

$$\nabla\cdot\mathbf{q} = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(h_2h_3u_1) + \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_2}(h_1h_3u_1) + \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_3}(h_1h_2u_1)$$

In spherical coordinates,

u_1, u_2 and u_3 are u_r, u_θ and u_ϕ (18)

Using equations (93) and (95), we obtain;

$$\begin{aligned}\nabla\cdot\mathbf{q} &= \frac{1}{r^2\sin\phi}\frac{\partial}{\partial r}(r^2\sin\phi u_r) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\theta}(ru_\theta) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}(r\sin\phi u_\phi)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\theta}(ru_\theta) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}(r\sin\phi u_\phi)\end{aligned} \tag{19}$$

Navier-Stokes equations

To express in spherical coordinates, the Navier-Stokes equation (60);

$$\rho\left(\frac{\partial\mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\mathbf{q}\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{q}$$

We first express the grad, laplacian, velocity and gravity in spherical coordinates.

In curvilinear coordinates, the laplacian from equation (67) is given by

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]$$

Replacing equation (92) and (94) into the above equation we obtain

$$\nabla^2 = \frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(\rho^2\sin\phi\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right] \tag{20}$$

We can also express g as

$$\mathbf{g} = g_r\mathbf{e_r} + g_\theta\mathbf{e_\theta} + g_\phi\mathbf{e}_\phi \tag{21}$$

From equation (43), in curvilinear coordinates, grad is given by;

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial}{\partial u_3}\mathbf{e_3}$$

In spherical coordinates the above equation is written as;

$$\nabla = \frac{\partial}{\partial r}\mathbf{e_r} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}\mathbf{e_\theta} + \frac{1}{r}\frac{\partial}{\partial\phi}\mathbf{e}_\phi \tag{22}$$

And finally we have;

$$\mathbf{q} = u_r\mathbf{e_r} + u_\theta\mathbf{e_\theta} + u_\phi\mathbf{e}_\phi \tag{23}$$

Now replacing equations, (92), (94) and (95) into equation (60) we obtain,

$$\rho\left(\frac{\partial}{\partial t}\left(u_r\mathbf{e_r} + u_\theta\mathbf{e_\theta} + u_\phi\mathbf{e}_\phi\right) + \left(\left(u_r\mathbf{e_r} + u_\theta\mathbf{e_\theta} + u_\phi\mathbf{e}_\phi\right).\left(\frac{\partial}{\partial r}\mathbf{e_r} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}\mathbf{e_\theta} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\mathbf{e}_\phi\right)\right)\left(u_r\mathbf{e_r} + u_\theta\mathbf{e_\theta} + u_\phi\mathbf{e}_\phi\right)\right)$$
$$= -\left(\frac{\partial}{\partial r}\mathbf{e_r} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}\mathbf{e_\theta} + \frac{1}{r}\frac{\partial}{\partial\phi}\mathbf{e}_\phi\right)p + \rho\left(g_r\mathbf{e_r} + g_\theta\mathbf{e_\theta} + g_\phi\mathbf{e}_\phi\right) +$$
$$\mu\left(\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right]\right)\left(u_r\mathbf{e_r} + u_\theta\mathbf{e_\theta} + u_\phi\mathbf{e}_\phi\right) \tag{24}$$

Expanding equation (96) we obtain

$$\left(\rho\frac{\partial}{\partial t}u_r\mathbf{e_r} + \rho\frac{\partial}{\partial t}u_\theta\mathbf{e_\theta} + \rho\frac{\partial}{\partial t}u_\phi\mathbf{e}_\phi\right) + \rho\left(u_r\frac{\partial}{\partial r}\mathbf{e_r} + \frac{u_\theta}{r\sin\phi}\frac{\partial}{\partial\theta}\mathbf{e_\theta} + \frac{u_\phi}{\rho}\frac{\partial}{\partial\phi}\mathbf{e}_\phi\right)\left(u_r\mathbf{e_r} + u_\theta\mathbf{e_\theta} + u_\phi\mathbf{e_j}\right)$$
$$= -\left(\frac{\partial p}{\partial r}\mathbf{e_r} + \frac{1}{r\sin\phi}\frac{\partial p}{\partial\theta}\mathbf{e_\theta} + \frac{1}{r}\frac{\partial p}{\partial\phi}\mathbf{e}_\phi\right) + \left(\rho g_r\mathbf{e_r} + \rho g_\theta\mathbf{e_\theta} + \rho g_\phi\mathbf{e}_\phi\right) + \tag{25}$$
$$\mu\left(\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right]\right)\left(u_r\mathbf{e_r} + u_\theta\mathbf{e_\theta} + u_\phi\mathbf{e}_\phi\right)$$

We then write equation (97) in component form, we first collect all the terms with $\mathbf{e_r}$;

$$\rho\frac{\partial}{\partial t}u_r + \rho\left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r\sin\phi}\frac{\partial}{\partial\theta} + \frac{u_\phi}{r}\frac{\partial}{\partial\phi}\right)u_r = -\frac{\partial p}{\partial r} + \rho g_r +$$
$$\mu\left(\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right]\right)u_r \tag{26}$$

Similarly terms containing $\mathbf{e_\theta}$ are,

$$\rho\frac{\partial}{\partial t}u_\theta e_\theta + \rho\left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r\sin\phi}\frac{\partial}{\partial\theta} + \frac{u_\phi}{\rho}\frac{\partial}{\partial\phi}\right)u_\theta = -\frac{1}{r\sin\phi}\frac{\partial p}{\partial\theta} + \rho g_\theta +$$
$$\mu\left(\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right]\right)u_\theta \tag{27}$$

And finally the terms containing e_ϕ

$$\rho\frac{\partial}{\partial t}u_\phi+\rho\left(u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r\sin\phi}\frac{\partial}{\partial\theta}+\frac{u_\phi}{\rho}\frac{\partial}{\partial\phi}\right)u_\phi=-\frac{1}{r}\frac{\partial p}{\partial\phi}+\rho g_\phi$$
$$+\mu\left(\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial}{\partial\phi}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right]\right)u_\phi \tag{28}$$

Equations; (98), (99) and (100) are the momentum equations in spherical coordinates.

CONCLUSION

In this paper we have outlined the expression of motion equations from vector form to spherical coordinates.

REFERENCES

1. Byerly, W. E. "Orthogonal Curvilinear Coördinates." §130 in An Elementary Treatise on Fourier's Series, and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics. New York: Dover, pp. 238-239, 1959.
2. Moon, P. and Spencer, D. E. Field Theory Handbook, Including Coordinate Systems, Differential Equations, and Their Solutions, 2nd ed. New York: Springer-Verlag, pp. 1-3, 1988.
3. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, Transport Phenomena, 2nd edition, Wiley: NY, 2002.
4. R. B. Bird, R. C. Armstrong, and O. Hassager, Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics,Wiley: NY, 1987.
5. Warsi, Z. U. A., \Conservation Form of the Navier-Stokes Equations inGeneral Nonsteady Coordinates," AIAA Journal, Vol. 19, No. 2, 1981.
6. Swungho Lee, \Three-Dimensional Incompressible Viscous Solutions Basedon the Physical Curvilinear Coordinate System," Dissertation, Departmentof Computational Engineering, Mississippi State University, December1997.[7] Peric, M., \Analysis of Pressure Velocity Coupling on Non-Staggered.
7. Grids," Numerical Heat Transfer, Part B, Vol. 17, pp 63{82, 1990.
8. Cho, M. J. and Chung M. K., \New Treatment of Nonorthogonal Termsin the Pressure-Correction Equation," Numerical Heat Transfer, Part B,Vol. 26, pp 133{145, 1994.
9. Baldwin, B. S. and Lomax, H., \Thin Layer Approximation and AlgebraicModel for Separated Turbulent Flows," AIAA 78{257, 1978.